Path Following of a Rolling Disk Using Throttle Only

Giuseppe Notarstefano, Maura Pasquotti and Ruggero Frezza

Abstract—In this paper we address the problem of controlling a disk, rolling on a horizontal plane, using only throttle as control input. The disk is supposed to follow an assigned path in the plane. The problem is difficult because of the high order of underactuation and of the instability of the system. The controller is based upon an internal manifold and a receding horizon technique. The lean angle reference trajectory is generated at each instant through a receding horizon algorithm and it is such that, if followed, the system tracks the assigned path with a bounded error.

Index Terms—Underactuated systems, nonholonomic vehicles, backstepping, receding horizon.

I. INTRODUCTION

Control of a rolling disk by throttle only represents a case study for a large class of mechanical systems. It is a high-order underactuated system [9] subject to nonholonomic constraints (pure rolling and no transversal slip). It is also a system with symmetry with respect to the horizontal plane coordinates, in fact its Lagrangian and constraints are invariant with respect to rotations and translations in the plane. Finally it has an unstable internal dynamics (the lean and side inclination angle dynamics).

To our knowledge, this type of problem has never been investigated so far. Control of a rolling disk has been studied only in the case of two control inputs, throttle and tilt moments, by Frangos and Yavin [2], [4]. They studied the path controllability of the disk and found a control action to track a given trajectory in the horizontal plane.

In our work we use a receding horizon technique in order to track a desired path in the plane with a bounded error. At each instant we find a feasible trajectory for the disk which originates from the current position and merges to the path at the time horizon. We call this trajectory “connecting contour”, see [5], [6]. Hence we use an internal manifold based control technique, inspired by Getz and Hedrick [7], [8], to find an equilibrium lean angle trajectory to track, in order to follow the connecting contour without the disk falling down.

The paper is organized as follows. In section 2 the nonholonomic model of the disk is presented together with a brief description of the method used to compute the dynamics. In section 3 we describe the control design for the lean angle tracking, based on a backstepping technique. In section 4 the receding horizon algorithm, used to track the desired path in the plane, is shown. We explain how the connecting contours are generated and how we construct the corresponding equilibrium lean trajectory. In section 5 some simulation results are shown and in section 6 we draw conclusions and some ideas of possible future work.

II. DYNAMICAL MODEL

The model we consider is a falling rolling disk subject to nonholonomic constraints. The disk, represented in fig.1, can roll in the horizontal plane without slipping and is subject to gravity acceleration. The configuration space for the disk is $Q = \mathbb{R}^2 \times S^3$ and is parameterized by the generalized coordinates $q = (x, y, \phi, \vartheta, \psi)$, where $(x, y)$ represent the position of the disk in the inertial frame, while $\phi, \vartheta$ and $\psi$ represent the angle between the plane of the disk and the vertical axis, the ”self-rotation” angle of the disk and the heading or yaw angle of the disk, respectively. The inertial frame is oriented according to the usual convention adopted in the automotive and aerospace fields, i.e. the $x$ axis is directed along the positive direction of motion of the vehicle ($\psi=0$), the $z$-axis points downward and the $y$-axis follows. The dynamics of the system can be written using a Lagrangian formalism. The Lagrangian for the unconstrained system is

\[ L = \frac{1}{2} \left[ (\dot{x} - \dot{\psi} R \sin \phi \cos \psi - \dot{\vartheta} \phi \cos \phi \sin \psi)^2 + \right. \\
+ (\dot{y} - \dot{R} \psi \sin \phi \sin \psi + \dot{\vartheta} \phi \cos \phi \cos \psi)^2] + \\
+ \frac{1}{2} m R^2 \dot{\vartheta}^2 \sin^2 \phi + \frac{1}{2} J (\dot{\vartheta}^2 + \dot{\psi}^2 \cos^2 \phi) + \\
\left. + \frac{1}{2} I (\ddot{\vartheta} + \dot{\psi} \sin \phi)^2 - mg R \cos \phi \right] \\
\]

where

- $m$ mass of the disk
- $I$ moment of inertia about the axis perpendicular to the plane of the disk
- $J$ moment of inertia about an axis in the plane of the disk
- $R$ disk radius.

The following constraints are imposed in order to avoid the wheel to slip laterally and longitudinally

\[ \dot{x} = -R \dot{\vartheta} \cos \psi \]
\[ \dot{y} = -R \dot{\vartheta} \sin \psi \]  

The set of generalized coordinates can be divided into $m$ constrained coordinates, denoted by $s$, and $n - m$
unconstrained, denoted by \( r, q = (s, r) \), where \( m \) is the number of constraints, see [1]. In our model \( s = (x, y) \) and \( r = (\phi, \psi, \vartheta) \). Now the “constrained” Lagrangian \( L_c \) is constructed by substituting the constraints (2) in (1), and the equation of motion can be written in terms of \( L_c \) in the following way:

\[
\frac{d}{dt} \frac{\partial L_c}{\partial \dot{r}} - \frac{\partial L_c}{\partial r} + A_{\alpha}^{a} \frac{\partial L_c}{\partial s^a} = \frac{\partial L}{\partial s^b} B_{\alpha \beta}^{b} \dot{r}^\beta
\]  

(3)

where

\[
A(r, s)[\dot{r} \quad \dot{s}]^T = 0
\]

is the velocity constraint and

\[
B_{\alpha \beta}^{b} = \frac{\partial A_{\alpha}^{b}}{\partial \dot{r}^{\beta}} - \frac{\partial A_{\beta}^{b}}{\partial r^{\alpha}} + A_{\alpha}^{a} \frac{\partial A_{\beta}^{b}}{\partial s^{a}} - A_{\beta}^{a} \frac{\partial A_{\alpha}^{b}}{\partial s^{a}},
\]

where the summation notation is used.

The dynamics of the unconstrained coordinates is thus obtained in terms of the constrained and unconstrained coordinates:

\[
M(s, r) \ddot{r} + C(s, r, \dot{r}) + G(s, r) = T.
\]  

(4)

This general form can be simplified in our model thanks to the symmetry of the system. For the rolling disk, in fact, the following holds:

**Fact:** \( L(s, r), A(s, r) \) are both independent of \( s \).

Taking this fact into account, after straightforward calculations, we obtain the dynamics of the unconstrained coordinates in the form:

\[
M(r) \ddot{r} + C(r, \dot{r}) + G(r) = T,
\]  

(5)

where

\[
M(r) = \begin{bmatrix}
  mR^2 + J & 0 & 0 \\
  0 & (mR^2 + I)s_\phi^2 + Jc_\phi^2 & 0 \\
  0 & (mR^2 + I)s_\phi & (mR^2 + I)
\end{bmatrix}
\]

\[
C(r, \dot{r}) = \begin{bmatrix}
  -(mR^2 + I)c_\phi \dot{\psi} - (mR^2 + I - J)s_\phi \dot{\psi} \\
  Jc_\phi \dot{\psi} + 2(mR^2 + I - J)s_\phi c_\phi \dot{\psi} \\
  2(mR^2 + I)c_\phi \dot{\psi}
\end{bmatrix}
\]

\[
G(r) = \begin{bmatrix}
  -mgR\sin \phi \\
  0 \\
  0
\end{bmatrix}
\]

and the torque vector

\[
T = \begin{bmatrix}
  0 \\
  0 \\
  \tau
\end{bmatrix}.
\]

See [1] for more details on the calculations of the dynamics.

**Remark 1:** The system has only one control input with three unconstrained variables, so it is an underactuated reduced system of order two [9].

**Remark 2:** The lean angle dynamics is not driven directly by \( \tau; \) instead it is coupled with the dynamics of \( \psi \) and \( \dot{\psi} \) by the centripetal acceleration term \( \psi \dot{\vartheta} \). Thus it can be considered the zero dynamics of the system w.r.t. the output \( \psi \dot{\vartheta} \). Being such dynamics unstable, the system is non-minimum phase.

**Remark 3:** If the disk starts from the vertical position \( (\phi = 0) \) with zero lean and yaw velocities \( (\dot{\phi} = 0 \text{ and } \psi = 0) \), the three scalar equations defined by (5) are completely decoupled and the control input only drives the \( \psi \) dynamics. Hence only a velocity regulation along the initial direction is possible, while the lateral dynamics is not controllable. The problem is thus well posed when we work far from the condition \( (\phi, \dot{\phi}, \psi) = (0, 0, 0) \).

Another critical situation occurs when the disk only traverses the vertical position \( (\phi = 0, \psi = 0, \dot{\phi} \neq 0) \). In that case the centripetal acceleration term in the \( \phi \) dynamics vanishes (so as the other Coriolis term and the gravity term, \( \dot{\psi} = 0 \)). Moreover the input \( \tau \) does not drive the \( \psi \) dynamics anymore. This corresponds in some sense to a loss of controllability as it will be shown in the lean angle control design section.

In order to avoid this condition we make the assumption that the curvature of the path has always the same sign. The problem of traversing the vertical position, in order to follow any type of path, will be the objective of future work.

For control purpose, we rewrite the system dynamics in state space form. Since the dynamics is invariant w.r.t. the yaw angle \( \psi \) and the self-rotation angle \( \vartheta \), they can be neglected and the state space vector results:

\[
x = [\dot{x}_1 \quad \dot{x}_2 \quad \dot{x}_3 \quad \dot{x}_4]^T = [\phi \quad \dot{\phi} \quad \psi \quad \dot{\psi}]^T.
\]

Multiplying both sides of (5) by \( M(s, r)^{-1} \) (the matrix of mass \( M(s, r) \) is certainly nonsingular for \( \phi \in (-\pi/2, \pi/2) \)) it follows:

\[
\dot{x}_1 = x_2
\]

\[
\dot{x}_2 = f_{01}(x_1) + f_{02}(x_1)x_3x_4 + f_{03}(x_1)x_2^2;
\]

\[
\dot{x}_3 = f_{01}(x_1)x_2x_4 + f_{02}(x_1)x_2x_3 + f_{03}(x_1)\tau;
\]

\[
\dot{x}_4 = f_{01}(x_1)x_2x_4 + f_{02}(x_1)x_2x_3 + f_{03}(x_1)\tau.
\]  

(6)
with  
\begin{align*}
  f_{\phi 1}(x_1) &= \frac{mgR}{mR^2 + J} \sin x_1; \\
  f_{\phi 2}(x_1) &= \frac{mR^2 + J}{mR^2 + J} \cos x_1; \\
  f_{\phi 3}(x_1) &= \frac{mR^2 + 1 - J}{mR^2 + J} \sin x_1 \cos x_1; \\
  f_{\psi 1}(x_1) &= -\frac{J}{J \cos x_1}; \\
  f_{\psi 2}(x_1) &= -\frac{J \sin x_1}{J \cos x_1} + \frac{2 \sin x_1}{\cos x_1}; \\
  f_{\psi 3}(x_1) &= -\frac{\sin x_1}{J \cos x_1}; \\
  f_{\theta 1}(x_1) &= \frac{I \sin x_1}{J \cos x_1}; \\
  f_{\theta 2}(x_1) &= \frac{I \sin^2 x_1}{J \cos x_1} - \frac{2 J \cos x_1}{mR^2 + J}; \\
  f_{\phi 3}(x_1) &= \frac{(mR^2 + 1) \sin^2 x_1 + J \cos^2 x_1 \sin \phi}{(mR^2 + 1) \cos x_1}.
\end{align*}

III. LEAN ANGLE CONTROL DESIGN

In this section we present a control strategy to track a preassigned "lean angle" trajectory. The leaning control is based on a backstepping control technique, see [3]. In order to apply it, the system has to be in the so called strict feedback form, that is

\begin{equation}
\begin{align*}
  \dot{z}_1 &= f_{\phi 1}(z_1) z_1 + f_{\phi 2}(z_1) z_2 + f_{\phi 3}(z_1) z_3 \\
  \dot{z}_2 &= f_{\theta 1}(z_2) z_1 + f_{\theta 2}(z_2) z_2 + f_{\theta 3}(z_2) z_3 \\
  \vdots \\
  \dot{z}_n &= f_{\phi 1}(z_n) z_1 + f_{\phi 2}(z_n) z_2 + f_{\phi 3}(z_n) z_3 + g(z_1, \ldots, z_n) u \\
  \dot{\bar{z}}_3 &= \frac{1}{f_{\psi 3}(\bar{z}_3)} (-f_{\psi 1}(\bar{z}_3) - k_{\phi}(\bar{z}_3 - \bar{z}_3) + \bar{\theta}_{ref} + \bar{\phi}_{ref}).
\end{align*}
\end{equation}

Even if the dynamics of the disk is not in such form, it can be reduced to (7) by way of an approximation on the lean angle dynamics and a change of coordinates.

First of all let us ignore, for control design purpose only, the term \( f_{\phi 3}(x_1) x_3^2 \) of the lean angle dynamics (it will be taken into account in the simulations). Then let us define the new set of coordinates

\begin{align*}
  \zeta_1 &= \phi; \\
  \zeta_2 &= \dot{\phi}; \\
  \zeta_3 &= \dot{\psi}; \\
  \zeta_4 &= \dot{\theta}.
\end{align*}

The new system dynamics becomes

\begin{align*}
  \dot{\zeta}_1 &= \zeta_2; \\
  \dot{\zeta}_2 &= f_{\phi 1}(\zeta_1) + f_{\phi 2}(\zeta_1) \zeta_3; \\
  \dot{\zeta}_3 &= f_{\psi 1}(\zeta_1) \zeta_2 \zeta_1^2 + f_{\psi 2}(\zeta_1) \zeta_2^2 + f_{\psi 3}(\zeta_1) \zeta_1 \zeta_3 + f_{\psi 4}(\zeta_1) \tau; \\
  \dot{\zeta}_4 &= f_{\phi 1}(\zeta_1) \zeta_2 \zeta_4 + f_{\phi 2}(\zeta_1) \zeta_2^2 + f_{\phi 3}(\zeta_1) \zeta_4.
\end{align*}

with  
\begin{align*}
  f_{\psi 1}(\zeta_1) &= -\frac{1}{J \cos \zeta_4}; \\
  f_{\psi 2}(\zeta_1) &= \frac{I \sin^2 \zeta_1}{J \cos \zeta_1}, \quad \frac{2}{\cos \zeta_1}, \quad \frac{1}{mR^2 + J} \cos \zeta_1; \\
  f_{\psi 3}(\zeta_1) &= \frac{2 \sin \zeta_1}{\cos \zeta_1}; \\
  f_{\psi 4}(\zeta_1) &= \frac{(mR^2 + 1) \sin^2 \zeta_1 + J \cos^2 \zeta_1 \sin \phi}{(mR^2 + 1) \cos \zeta_1}.
\end{align*}

The term \( f_{\psi 4}(\zeta_1) \) in (9) vanishes when both \( \zeta_1 = \phi \) and \( \zeta_3 = \psi \) go to zero, thus making the \( \zeta_3 \) dynamics uncontrollable. This condition corresponds to the situation, mentioned in section 2, of the disk traversing the vertical position. Hence such condition is in somehow pathological in that it makes the system (partially) uncontrollable. Nevertheless the singularity is avoided under the assumption that the curvature of the path has always the same sign.

The subsystem given by the first three equations of (8) is in the form (7). As velocity in the path following is not a concern, the dynamics of \( \zeta_4 = \theta \) can be ignored. Observe that if \( \zeta_3 \) remains bounded the only case \( \zeta_4 \) could diverge is when \( \psi \) goes to zero. But this cannot happen under the above assumption on the path curvature.

Now we choose \( \zeta_3 \) as virtual control input to follow the lean angle reference. Using dynamic inversion for the lean angle dynamics we have:

\begin{equation}
\begin{align*}
  \bar{\zeta}_3 &= \frac{1}{f_{\psi 3}(\bar{z}_3)} (-f_{\psi 1}(\bar{z}_3) - k_{\phi}(\bar{z}_3 - \bar{\phi}_{ref}) + \bar{\phi}_{ref} + \bar{\theta}_{ref}).
\end{align*}
\end{equation}

At this point we can make \( \zeta_3 \) follow \( \bar{\zeta}_3 \), so we obtain the following throttle control:

\begin{equation}
\begin{align*}
  \tau &= \frac{1}{f_{\psi 4}(\zeta_1)} \left( f_{\phi 3}(\zeta_1) - \frac{\zeta_4}{\bar{z}_3} - \bar{\zeta}_3 \right) \quad (11)
\end{align*}
\end{equation}

where

\begin{align*}
  f_{\phi 3}(\zeta) &= f_{\phi 1}(\zeta_1) \zeta_2 \zeta_4 + f_{\psi 2}(\zeta_1) \zeta_2^2 + f_{\psi 3}(\zeta_1) \zeta_1 \zeta_3 + f_{\psi 4}(\zeta_1) \tau; \\
  \text{and}
\end{align*}

\begin{align*}
  \bar{\zeta}_3 &= \frac{mR^2 + J}{(mR^2 + J) \cos \zeta_1} \left( \frac{-mgR \zeta_2 \sin \zeta_1}{mR^2 + J} - k_{\phi}(\zeta_2 - \bar{\phi}_{ref}) + \bar{\phi}_{ref} + \bar{\theta}_{ref} \right) + \\
  &\quad + \frac{(mR^2 + J) \zeta_2 \sin \zeta_1}{(mR^2 + J) \cos \zeta_1} \left( \frac{-mgR \zeta_2 \sin \zeta_1}{mR^2 + J} - k_{\phi}(\zeta_2 - \bar{\phi}_{ref}) + \bar{\phi}_{ref} + \bar{\theta}_{ref} \right).
\end{align*}

The resulting closed-loop system is:
\[\begin{align*}
\dot{\zeta}_1 &= \zeta_2; \\
\dot{\zeta}_2 &= -k_\phi (\zeta_1 - \phi_{\text{ref}}) - k_\phi (\zeta_2 - \dot{\phi}_{\text{ref}}) + \ddot{\phi}_{\text{ref}} + f_{\phi 2}(\zeta_1)\dot{\zeta}_1; \\
\dot{\zeta}_3 &= -k_\zeta (\zeta_3 - \zeta_3) + \ddot{\zeta}_3
\end{align*}\]

where

\[\ddot{\zeta}_3 = \zeta_3 - \zeta_3.\]

From the last equation \(\dot{\zeta}_3\) converges to \(\zeta_3\), that is \(\dot{\zeta}_3\) goes to zero. Since the \((\zeta_1, \zeta_2)\) subsystem is a zero-GAS (Globally Asymptotically Stable) linear system, it is ISS (Input to State Stable) [3], hence \(\zeta_1\) converges to the desired value.

IV. RECEIVING HORIZON CONTROL

Our objective is to follow a trajectory in the plane that is close, in some sense, to a desired path. Since we handle a control only, we cannot assign the velocity on the path. The lean angle control we designed in the previous section allows to track a lean angle trajectory, but it does not ensure that the disk stay close to the generated trajectories of the same lean angle trajectory could correspond to different respect to \(x\) (hence the lean angle dynamics) is, in fact, invariant with \(x\), \(y\) and \(\psi\), hence the same lean trajectory could correspond to the same path rotated or translated in the plane. Moreover, in dependence of the rotation velocity \(\dot{\theta}\), the same lean angle trajectory could correspond to different trajectories of \(\dot{\psi}\) and hence to different curvatures of the path. In fact the dynamics of \(\dot{\phi}\)

\[\dot{\phi} = f_{\phi 1}(\phi) + f_{\phi 2}(\phi)\dot{\theta} + f_{\phi 3}(\phi)\dot{\psi}^2 \tag{13}\]

is invariant with respect to every \(\dot{\psi}\) and \(\dot{\theta}\) such that

\[f_{\phi 2}(\phi)\dot{\psi} \dot{\theta} + f_{\phi 3}(\phi)\dot{\psi}^2 = a(\phi(t)), \tag{14}\]

with \(a(\phi(t))\) satisfying (13). Since

\[\dot{\psi} = -\sigma(s(t))R\dot{\theta}(t), \tag{15}\]

where

\[s(t) = \int_0^t R\dot{\theta}(\tau)d\tau,\]

different possible paths are compatible with (14), (15). Our idea is to use a predictive control strategy to converge to the exact path in the manifold of the paths compatible with the same lean angle trajectory. We generate a trajectory in the plane which is feasible for the disk, starting from the current position and orientation and merging to the path at look ahead distance \(D\), fig.2. Then we find a \(\phi\) trajectory that, if followed, makes the disk stay close to the generated path and, at this point, we use the previous closed-loop control action.

The control strategy can be summarized as follows:

1. At time \(t\) the contour and its first \(p\) derivatives are computed at look-ahead distance \(D\);
2. A feasible trajectory which connects at look-ahead distance \(D\) to the path is generated. We call the trajectory connecting contour;
3. The angular velocity trajectory \(\dot{\psi}_c(t)\) associated to the connecting contour is computed, assuming knowledge of the velocity of the disk along the path;
4. An equilibrium lean angle trajectory compatible with \(\dot{\psi}_c(t)\) (and with the velocity of the disk) is generated;
5. The lean angle control action of section III is applied to track the equilibrium \(\phi\) trajectory;
6. The previous steps are repeated recursively.

So far we can see that this strategy is predictive and closed loop, but we need to specify how to make it causal, since we suppose to know the velocity of the disk in the future (and that velocity depends on the control action which is what we want to compute). Observe, however, that the actual trajectory of the disk is not any single connecting contour but, rather, the envelope of the connecting contours.

Assume that, in the moving frame attached to the disk, the path can be represented locally as:

\[y = \gamma(x,t) \quad x \in [0, D]. \tag{16}\]

The connecting contour should satisfy two conditions: it should be feasible for the system and it should satisfy some optimality criterion. The simplest criterion is to design a trajectory which intersects the path with the proper derivatives. Therefore the connecting contour \(\gamma_c(x,t)\) must satisfy the following boundary conditions:

\[
\begin{bmatrix}
\gamma_c(0,t) \\
\frac{\partial \gamma_c}{\partial x}(0,t)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}

\begin{bmatrix}
\gamma_c(D,t) \\
\frac{\partial \gamma_c}{\partial x}(D,t)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}

\begin{bmatrix}
\gamma(D,t) \\
\frac{\partial \gamma}{\partial x}(D,t)
\end{bmatrix}
\tag{17}
\]

Among all possible connecting contours, we chose polynomials (which have the advantage of minimizing the overall curvature) and set \(p = 2\) for the simulations.
If we now suppose to know the velocity of the disk along the connecting contour $\dot{\psi}_c(t)$, we can compute the angular velocity $\dot{\psi}_c(t)$ associated to the path as:

$$\dot{\psi}_c = -\sigma_c(s(t)) R \dot{\psi}_c(t). \tag{18}$$

Having determined the value of $\dot{\psi}_c(t)$, we seek a bounded lean angle trajectory compatible with $\psi_c(t)$ and $\dot{\psi}_c(t)$. This trajectory is a time varying equilibrium for the lean angle dynamics and is obtained by imposing:

$$0 = \frac{1}{mR^2+I} (mgR \sin \phi_c + (mR^2 + I) \dot{\psi}_c \dot{\psi}_c \cos \phi_c + (mR^2 + I - J) \dot{\psi}_c^2) \tag{19}$$

At this point, observe that the receding horizon algorithm involves the control action is applied only for the first time step. Since the lean angle control action is not predictive we only need the value of $\phi_c(t)$ at the current time, which we call $\bar{t}$. This implies that we need the value of $\phi_c(t)$ at $\bar{t}$ only. Hence we can use its current value $\dot{\psi}_c(\bar{t})$. This means we suppose to follow the path at the current velocity, at least in the first time interval. In the lean angle control we would also need the value of $\phi_c(\bar{t})$, $\dot{\phi}_c(\bar{t})$ and $\phi_c^{(3)}(\bar{t})$. Since in the control design we made an approximation on the $\phi$ dynamics and we are not interested in an exact tracking, we can neglect the last two feed-forward terms. On the contrary $\phi_c(\bar{t})$ can be computed by considering the approximated value of $\phi_c(\bar{t})$

$$\phi_c(\bar{t}) = -\arctan \left( \frac{(mR^2 + I) \dot{\psi}_c(\bar{t}) \dot{\psi}_c(\bar{t})}{mgR} \right) \tag{20}$$

as:

$$\dot{\phi}_c(\bar{t}) = -\frac{mgR(mR^2 + I) \dot{\psi}_c(\bar{t}) \dot{\psi}_c(\bar{t}) + \dot{\psi}_c(\bar{t}) \dot{\psi}_c(\bar{t})}{mgR}\phi_c(\bar{t}) + \phi_c(\bar{t}) \dot{\psi}_c(\bar{t}) \dot{\psi}_c(\bar{t}) \right) \right) \tag{21}$$

where $\ddot{\phi}_c(\bar{t})$ is known and

$$\ddot{\psi}_c(\bar{t}) = \frac{\sigma_c(s(\bar{t})) R^2 \ddot{\psi}_c(\bar{t}) - \sigma_c(s(\bar{t})) R \ddot{\psi}_c(\bar{t})}{\sigma_c(s(\bar{t})) R}. \tag{22}$$

A rigorous proof of the convergence of the receding horizon algorithm to the desired lean trajectory is missing and it will be the objective of future work. However the control strategy has been tested by performing simulations on several maneuvers.

V. SIMULATION RESULTS

In this section we show the simulations results of the controlled disk following a circle and a parabolic path. The disk parameters used in the simulations are: $m = 10\, [Kg]$, $R = 0.3\, [m]$, $I = 0.4\, [Kg \cdot m^2]$, $J = 0.2\, [Kg \cdot m^2]$ and $g = 9.8\, [m/s^2]$.

Figure 3 shows the disk tracking a circle path. The disk starts from the condition $x(0) = -5.0\, [m]$, $y(0) = 0.5\, [m]$, $\dot{x}(0) = -45\, [deg]$, $\dot{y}(0) = 0\, [deg/s]$, $\psi(0) = 65\, [deg]$, $\dot{\psi}(0) = -25\, [deg/s]$, $\psi^2(0) = -10\pi / 2\, [rad/sec]$, recovers to the right inclination and converges to the circle. In fig. 3 (a) the resulting path in the ground plane is shown. Figure 3 (b) shows the lean angle trajectory, compared with the equilibrium trajectory, and its derivative. In fig. 3(c) the yaw trajectory and its derivative are shown and finally the control input and the rotation velocity are depicted in fig. 3(d).

Figure 4 shows the same graphs in the case of a parabolic path. The initial condition is set to $x(0) = 5\, [m]$, $y(0) = -2.1\, [m]$, $\phi(0) = 50\, [deg]$, $\dot{\phi}(0) = 0\, [deg/s]$, $\psi(0) = 10\, [deg]$, $\dot{\psi}(0) = 25\, [deg/s]$, $\dot{\psi}(0) = -10\pi / 2\, [rad/sec]$.

VI. CONCLUSIONS AND FUTURE WORK

The problem of controlling a rolling disk has been studied. The disk is made follow an assigned path in the plane by using throttle only. We used a receding horizon control strategy to generate, at each instant, a feasible trajectory in the plane starting from the current position of the disk and converging to the path at a look-ahead distance. A bounded-lean equilibrium trajectory, compatible with the trajectory in the plane, is computed and is tracked by means of a backstepping control strategy.

In order to avoid singularities we made the assumption that the disk does not traverse the vertical position. The aims of our future work are, on one hand, to solve the same problem without introducing the previous restriction and, on the other hand, to generalize the control technique for the control of several non-minimum phase and high-order underactuated systems.

REFERENCES

Fig. 3. Following of a circular path

Fig. 4. Following of a parabolic path