Abstract: In this paper we provide a new strategy to explore feasible trajectories of nonlinear systems, that is to find curves that satisfy the dynamics as well as pointwise state-input constraints. This strategy is interesting itself in understanding the behavior of the system especially in critical conditions, and represents a useful tool that can be used to perform trajectory tracking in presence of constraints. The strategy is based on a novel optimization technique, introduced by Hauser, to find regularized solutions for point-wise constrained optimization of trajectory functionals. The strategy is applied to the PVTOL, a simplified model of a real aircraft that captures the main features and challenges of the real model.

Keywords: Nonlinear optimal control, constrained optimization, PVTOL

1. INTRODUCTION

Output tracking is a challenging task in the control of nonlinear systems. It has interesting practical applications in several fields as aerospace, robotics and automotive. Given an output reference trajectory, the control objective is to find a feedback law such that the output asymptotically tracks the reference trajectory (maintaining internal stability). A preliminary step or a different point of view in the solution of this problem is the exploration of the trajectory manifold of the system, that is the characterization of all output trajectories that can be tracked and the parametrization of the state-input trajectories with respect to the output ones. Having in mind engineering applications, there is an important aspect to take into account in the solution of exploration and tracking, that is the presence of constraints in the system. Such constraints, we will call them operating conditions, may arise from diverse causes such as physical bounds on the states and the inputs or presence of regions where the model is valid or where “bad” phenomena (like loss of controllability) are ensured not to appear. This means that not only we look for trajectories (curves satisfying the dynamics), but furthermore we need feasible trajectories, that is trajectories lying in a region where the operating conditions are satisfied. In this paper we solve this problem for the PVTOL aircraft under input constraints.
The PVTOL is a simplified model introduced in (Hauser et al., 1992) in order to capture the lateral non-minimum phase behavior of a real aircraft. This model has been widely studied in the literature for its characteristic of combining important features of nonlinear systems with “tractable” equations. Furthermore the dynamics of many other mechanical systems can be rewritten in a similar fashion, e.g. the cart-pole system, the pendubot (Spong and Block, 1995), the bicycle model (Getz and Marsden, 1995), (Hauser et al., 2004) and the longitudinal dynamics of a real aircraft.

In this section we present the PVTOL dynamics, its flatness property and its non-minimum phase behavior of a real aircraft.

We attack the problem of exploring feasible trajectories by using optimal control. The strategy consists of the following steps. Given a desired output trajectory, first a state-input trajectory consistent with the output is computed as in (Notarstefano et al., 2005). Then, a feasible trajectory, close in the $L_2$ norm to the desired one, is computed by solving a regularized version of an optimal control problem with pointwise constraints. This regularization is based on the introduction of a parameterized barrier functional, to constraints. This regularization idea for solving constrained optimal control was introduced in (Hauser and Saccon, 2006). This effective technique is just based on the regularization of a trajectory functional optimization with pointwise constraints and then on its solution by means of a projection operator based Newton method (Hauser, 2002). We demonstrate the effectiveness of the proposed strategy. In particular we provide results showing that for suitable choice of the constraints and the design parameters, the solution of the regularized optimal control problem exists and satisfies second order sufficiency conditions. In simulations we show that for the PVTOL example the solution behaves very well even in presence of relatively tight constraints.

2. PVTOL MODELING

In this section we present the PVTOL dynamics, its flatness property and its non-minimum phase nature with respect to outputs.

In (Hauser et al., 1992) the PVTOL aircraft model was introduced. Using standard aeronautic conventions the equations of motion are given by

\[
\begin{align*}
\ddot{y} &= u_1 \sin \varphi - \epsilon_{PVTOL} u_2 \cos \varphi \\
\ddot{z} &= -u_1 \cos \varphi - \epsilon_{PVTOL} u_2 \sin \varphi + g \\
\dot{\varphi} &= u_2,
\end{align*}
\]

The aircraft state is given by the position $(y, z)$ of the center of gravity, the roll angle $\varphi$ and the respective velocities $\dot{y}, \dot{z}$ and $\dot{\varphi}$. The control inputs $u_1$ and $u_2$ are respectively the vertical thrust force and the rolling moment. The rolling moment $u_2$ generates also a lateral force (the lift forces are not perpendicular to the wings) and $\epsilon_{PVTOL}$ is the coupling coefficient. Finally $g$ is the gravity acceleration. In Figure 1 the PVTOL aircraft with the reference system and the inputs is shown.

![PVTOL aircraft](image)

Fig. 1. PVTOL aircraft.

(Hauser et al., 1992) the PVTOL was shown to be input-output linearizable when $\epsilon_{PVTOL} = 0$, while in (P.Martin et al., 1996) it was shown that, when $\epsilon_{PVTOL}$ is not zero, suitable outputs (flat outputs) may be found, such that the system can be feedback linearized by means of dynamic extension. Using as flat outputs

\[
\begin{align*}
y &= y + \epsilon_{PVTOL} \sin(\varphi) \\
z &= z + \epsilon_{PVTOL} \cos(\varphi)
\end{align*}
\]

it can be shown, after some straightforward calculations, that for all $\dot{\varphi}$ and $u_1$ such that $\tilde{u}_1 \neq 0$, $\tilde{u}_1 = u_1 - \epsilon_{PVTOL} \dot{\varphi}^2$, the system is feedback linearizable and in fact equivalent to the two dimensional forth order integrator $\dot{y}$ with suitable expressions for $v_1$ and $v_2$.

Remark 2.1. For $\epsilon_{PVTOL} = 0$ the condition $\tilde{u}_1 \neq 0$ reduces to $u_1 \neq 0$. This means that if $u_1$ is positive (physical constraint) and bounded away from zero, the system can be always feedback linearized despite of the value of the states.

The PVTOL has relative degree $r = [2, 2]$. Posing

\[
\begin{bmatrix}
u_1 \\ u_2
\end{bmatrix}
= \begin{bmatrix}
\sin \varphi & -\cos \varphi \\
-\cos \varphi & -\sin \varphi
\end{bmatrix}
\begin{bmatrix}
0 \\ -g
\end{bmatrix} +
\begin{bmatrix}
v_1 \\ v_2
\end{bmatrix}
\]

the dynamics of the system become

\[
\begin{align*}
\dot{y} &= v_1 \\
\dot{z} &= v_2 \\
\dot{\varphi} &= \frac{(g - v_2)}{\epsilon_{PVTOL}} \sin \varphi - \frac{v_1}{\epsilon_{PVTOL}} \cos \varphi.
\end{align*}
\]

As expected, the output dynamics are linear, however the zero dynamics is unstable. Equation (2) is the dynamics of a Driven Pushed Pendulum. In this sense the PVTOL can be seen as a general case of many other mechanical systems sharing a pendulum-like dynamics.
An important role in the study of the trajectory manifold of the PVTOL is played by the “quasi trajectory” that (with some abuse of notation) we call quasi static trajectory. It is defined as:
\[ \tan \varphi_{qs}(t) = \frac{\dot{y}(t)}{(g - \ddot{z}(t))} \]
and it is a curve built pretending that at each instant the roll angle assumes the equilibrium value as \( \dot{y}(t) \) and \( \ddot{z}(t) \) were constant. It is worth noting that the quasi static roll trajectory is exactly the roll trajectory for the model with \( \epsilon_{PVTOL} = 0 \). This provides a further motivation, in the next section, to search a roll trajectory “close” to the quasi static approximation.

3. DICHOTOMY AND EXISTENCE OF A BOUNDED ROLL TRAJECTORY

We recall the results presented in (Notarstefano et al., 2005). Given a \( C^1 \) output trajectory (of the center of gravity) on the infinite time interval \( \mathbb{R} \), satisfying, for a given \( a_{\text{max}} > 0 \),
\[ 0 < (g - v_2(t))^2 + v_1(t)^2 < a_{\text{max}}, \forall t \in \mathbb{R}, \]
there exists a bounded roll trajectory consistent with the output one. It is worth noting that, starting with \( u_1(0) > 0 \), the above condition is equivalent to \( 0 < u_1(t) < a_{\text{max}}, \forall t \in \mathbb{R} \). The proof of existence is based on the presence of a dichotomy in the linearization of the pendulum dynamics about the vertical position. The bounded trajectory is proven to exist as a fixed point of a contraction mapping (Hauser et al., 2005).

The bounded roll trajectory is computed, on a finite horizon \([0, T]\), by means of a dynamic embedding technique, introduced in (Hauser et al., 2004). The method consists of embedding the original roll dynamics in the driven system
\[ \ddot{\varphi} = \frac{(g - v_2)}{\epsilon_{PVTOL}} \sin \varphi - \frac{v_1}{\epsilon_{PVTOL}} \cos \varphi + u_{\text{ext}}, \quad (3) \]
where \( u_{\text{ext}} \) is a fictitious input used to drive the system along any desired admissible trajectory. If the accelerations of the center of gravity vary slowly, we can imagine the roll trajectory to be close to the quasi-static one. Hence, the quasi-static may be used as an initial guess to find the actual trajectory. If we rewrite (3) in state space form as \( \dot{x}_\phi = f_\phi(t, x_\phi, u_{\text{ext}}) \), where \( x_\phi = (\varphi, \dot{\varphi}) \) and \( x_{qs} = (\varphi_{qs}, \dot{\varphi}_{qs}) \), the following optimization problem may be posed:

Minimize \[ \frac{1}{2} \int_0^T \|x_\phi(\tau) - x_{qs}(\tau)\|^2_Q + \rho \|u_{\text{ext}}(\tau)\|^2 d\tau \]
subject to \( \dot{x}_\phi = f_\phi(t, x_\phi, u_{\text{ext}}) \),

where \( Q, \rho \) and \( P \) are weighting parameters. Using a high weight \( \rho \) for the input, we may obtain a trajectory arbitrarily close to the bounded roll trajectory we are looking for. The optimization problem is solved by using the projection operator based Newton method described in the next session.

4. FINDING FEASIBLE TRAJECTORIES BY USE OF OPTIMAL CONTROL

In this section we describe the strategy to perform feasible trajectories exploration and provide existence results for suitable operating conditions. We consider control systems \( \Sigma \) of the form
\[ \begin{align*}
\dot{x} &= f(x, u), \\
y &= p(x)
\end{align*} \quad (4) \]
where \( f(x, u) \) and \( p(x) \) are \( C^r \) mappings, \( r \geq 2 \), \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) the input and \( y \in \mathbb{R}^p \) is the objective output, used to define the task. We say that a bounded curve \( \eta = (x(\cdot), u(\cdot)) \) is a (state-input) trajectory of \( \Sigma \) if
\[ \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad \forall t \geq 0. \]

Given a desired output curve (candidate output trajectory) \( y_d(\cdot) \), we say that \( \xi_t = (x_d(\cdot), u_d(\cdot)) \) is a lifted trajectory of \( y_d(\cdot) \) if \( \dot{x}_d = f(x_d, u_d) \) (\( \xi_t \) is a trajectory) and \( y_d(\cdot) = p(x_d(\cdot)) \). If such \( \xi_t \) exists, we call \( y_d(\cdot) \) an output trajectory.

Although we will be mostly interested in trajectories on the finite horizon \([0, T]\), it is often useful to consider a finite length trajectory as a portion of one of infinite extent. Since \( \Sigma \) may be inherently unstable, we take a trajectory tracking approach. To this end, suppose that \( \xi(t) = (\alpha(t), \mu(t)), \quad t \geq 0 \), is a bounded curve (e.g., an approximate trajectory of \( \Sigma \)) and let \( \eta(t) = (x(t), u(t)), \quad t \geq 0 \), be the (state-input) trajectory of \( \Sigma \) determined by the nonlinear feedback system
\[ \dot{x}(t) = f(x(t), u(t)), \quad x(0) = \alpha(0), \]
\[ u(t) = \mu(t) + K(t)(\alpha(t) - x(t)) \]
Under certain conditions on \( f \) and \( K \), this feedback system defines a continuous, nonlinear projection operator
\[ \mathcal{P} := (\alpha, \mu) \mapsto \eta = (x, u), \]
that is, independent of \( K \), if \( \xi \) is a trajectory of \( \Sigma \), then \( \xi \) is a fixed point of \( \mathcal{P} \). Now suppose that \( \xi \) is a trajectory of \( \Sigma \) of infinite extent and that \( K \) is bounded and such that the above feedback exponentially stabilizes \( \xi_0 \). Then \( \mathcal{P} \) is well defined on an \( L_\infty \) neighborhood of \( \xi_0 \) and is \( C^r \) on its domain (including an open neighborhood of \( \xi_0 \)) whenever \( f \) is (Hauser, 2002). (By differentiable, we mean Fréchet differentiable with respect to the \( L_\infty \) norm.) Exponential stability plays an important role in making this operator continuous. Further properties of \( \mathcal{P} \) can be used to show that the set of exponentially stabilizable trajectories of \( \Sigma, T \), is a Banach manifold (Hauser and Meyer, 1998).
4.1 Projection operator based Newton method

Let $X$ denote the closed subspace of $L^2_{\text{ref}}[0,T]$ of curves $\zeta = (\beta, \nu)$ with continuous $\beta, \beta'(0) = 0$, and bounded $\nu$. Equipped with the norm $\|\zeta\|_X = \|\zeta\|_{L^\infty} \cdot X$ is a Banach space. Define $\pi_1 := [I \ 0]$ and $\pi_2 := [0 \ I]$ so that $\beta = \pi_1 \zeta$ and $\nu = \pi_2 \zeta$. Trajectories of $\Sigma$ through $x_0$ belong to the affine subspace $\tilde{X} := (x_0, 0) + X$. Defining the functional

$$h(\zeta) := \int_0^T l(\tau, \zeta(\tau))d\tau + m(\pi_1 \zeta(T))$$

for curves $\zeta = (\alpha, \mu) \in \tilde{X}$, we see that the unconstrained optimal control problem is equivalent to the constrained optimization problems

$$\min_{\zeta \in T} h(\zeta) = \min_{\zeta \in \mathcal{P}(\zeta)} h(\zeta)$$

where the constraint set $T$ is a Banach submanifold of $\tilde{X}$. Here $l(t, x, u)$ is $C^2$ or better in $x$ and $u$, convex in $u$, and $C^1$ or better in $t$, and $m(x)$ is $C^2$ or better in $x$. Defining

$$g(\zeta) := h(\mathcal{P}(\zeta))$$

for $\zeta \in \mathcal{U} \subset \tilde{X}$ with $\mathcal{P}(\mathcal{U}) \subset \mathcal{U} \subset \text{dom} \mathcal{P}$, we see that the optimization problems

$$\min_{\zeta \in T} h(\zeta) \text{ and } \min_{\zeta \in \mathcal{U}} g(\zeta)$$

are equivalent in the following sense. If $\zeta^* \in T \cap \mathcal{U}$ is a constrained local minimum of $h$, then it is an unconstrained local minimum of $g$. If $\zeta^* \in \mathcal{U}$ is an unconstrained local minimum of $g$ in $\mathcal{U}$, then $\zeta^* = \mathcal{P}(\zeta^*)$ is a constrained local minimum of $T$. This observation is the basis for the development of a family of quasi-Newton descent methods for the optimization of $h$ over $T$. The projection operator $\mathcal{P}$ provides a convenient parametrization of the trajectories in the neighborhood of a given trajectory. Indeed, the tangent space $T_{\zeta^*} T$ of bounded trajectories of the linearization of $\dot{x} = f(x, u)$ about $\zeta \in T$ can be used to parameterize all nearby trajectories (Hauser and Meyer, 1998). That is, given $\zeta \in T$, there is $\epsilon > 0$ such that, for each $\eta \in T$ with $\|\eta - \zeta\| < \epsilon$ there is a unique $\zeta^* \in T_{\zeta^*} T$ such that $\eta = \mathcal{P}(\zeta^*)$. Note also that $\zeta \mapsto D\mathcal{P}(\zeta) \cdot \zeta$ is the bounded linear projection operator defined by linearizing (4) about $\zeta$ and that $\zeta^* \in T_{\zeta^*} T$ if and only if $\zeta^* = D\mathcal{P}(\zeta^*) \cdot \zeta$. The projection operator based Newton method is the following.

**Algorithm** (projection operator Newton method)

Given initial trajectory $\zeta_0 \in T$

For $i = 0, 1, 2, \ldots$

- Design $K$ defining $\mathcal{P}$ about $\zeta_i$ search direction
  $$\zeta_i = \arg \min_{\zeta \in T_{\zeta_i} T} Dg(\zeta_i) \cdot \zeta + \frac{1}{2} D^2 g(\zeta_i)(\zeta, \zeta)$$
- Step size $\gamma_i = \arg \min_{\gamma \in [0, 1]} g(\zeta_i + \gamma \zeta_i)$
- Project $\zeta_{i+1} = \mathcal{P}(\zeta_i + \gamma_i \zeta_i)$.

4.2 Feasible trajectories exploration

In the previous section, we have seen a strategy to compute an unconstrained trajectory of the system given a desired output. However, it may happen that the lifted trajectory does not satisfy the operating conditions. Therefore we can informally state our objective.

**Objective** [informal description] Design a tool that, given a desired output trajectory, provides a feasible trajectory (state-input), that meets the operating conditions and is such that its output part is “close” to the desired one.

Clearly, regarding the objective, we need to define formally the notion of “closeness” (between the desired and the feasible outputs) and the notion of feasible trajectory. In this paper we use a weighted $L_2$ norm between the desired trajectory (lifted from the desired output) and the feasible one. Given a desired trajectory $\xi_d$ (lifted from a desired output), we set the distance of a trajectory $\xi$ from $\xi_d$ to

$$h(\xi) = \int_0^T \frac{1}{2} \|x(\tau) - x_d(\tau)\|^2_Q + \frac{1}{2} \|u(\tau) - u_d(\tau)\|^2_R + \frac{1}{2} \|T(\tau) - x_d(\tau)\|^2_{P_t}$$

where $Q$, $R$ and $P_t$ are positive definite matrices. The reason we decide to fully penalize the state and the input will be clear in the following when we state the optimal control problem. The operating conditions are taken into account defining the following region in the state-input space

$$\mathcal{X}U_\rho = \{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid c_j(x, u; \rho) \leq 0, \rho > 0, j = 0, 1, \ldots, k \}$$

where each $c_j(x, u; \rho)$ is $C^2$ or better in $x$ and $u$ and varies smoothly with the parameter $\rho$, and the following assumption holds.

**Operating Conditions Assumption.** The feasible region satisfies the following properties:

(i) for every $\rho > 0$, $\mathcal{X}U_\rho$, the interior of $\mathcal{X}U_\rho$, is a nonempty connected set.

(ii) for every $0 < p_1 < p_2$, $\mathcal{X}U_{p_1} \subset \mathcal{X}U_{p_2}$.

(iii) the projection of $\mathcal{X}U_\rho$ on the input space $\pi_u \mathcal{X}U_\rho = \{ u \in \mathbb{R}^m \mid (x, u) \in \mathcal{X}U_\rho \forall \text{ fixed } x \}$ is convex.

(iv) for every desired output trajectory $y_d(\cdot)$, the lifted trajectory $\xi_d = (x_d(\cdot), u_d(\cdot))$ (if it exists) is such that $\exists \rho_0 > 0$ such that $(x_d(t), u_d(t)) \in \mathcal{X}U_{\rho_0}$ for every $t \in [0, T]$. ☐

Roughly speaking the operating conditions describe a region where the space and the input must lie at every instant. For theoretical purpose we parameterize this region with a scaling factor $\rho > 0$ that allows us to expand and shrink the region in such a way that we can always entirely include the desired trajectory in it.
The last ingredient to introduce is the barrier functional associated with the region \( \mathcal{XU}_\rho \)

\[
b_{k_e, \rho}(\xi) = \int_0^T \sum_j \beta_{k_e}(-c_j(\tau, \alpha(\tau), \mu(\tau); \rho))
\]

where \( z \mapsto \beta_{k_e}(z) \) is defined as

\[
\beta_{k_e}(z) = \begin{cases} 
  -\log z & z > \delta_c \\
  \frac{k-1}{k} \left( \frac{z - k\delta_c}{(k-1)\delta_c} \right)^k - 1 - \log \delta_c & z \leq \delta_c.
\end{cases}
\]

With this definition in hand we can define the modified cost functional

\[
h_{c_e, \rho}(\xi) = h(\xi) + \epsilon_c b_{k_e, \rho}(\xi).
\]

and consistently

\[
g_{c_e, \rho}(\xi) = h_{c_e, \rho}(\mathcal{P}(\xi)).
\]

This barrier functional is the key element of the new optimization technique introduced in (Hauser and Saccon, 2006). In fact the \( C^2 \) function \( \beta_{k_e}(\cdot) \) retains many of the important properties of the usual \( \log \) barrier function \( z \mapsto \log(z) \) (used in constrained finite dimension optimization), while expanding the domain of finite values from \((0, \infty)\) to \((-\infty, +\infty)\). This ensures that the functional \( h_{c_e, \rho}(\xi) \) can be evaluated on any curve \( \xi \) in \( X \) and not only on feasible ones.

The exploration strategy is described in the table.

\[
\begin{array}{ll}
given \quad y_d(\cdot) \in C^r, r > 1 \quad \text{and} \quad \mathcal{XU}_\rho, \rho > 0 \quad \text{(set of operating conditions)}
\end{array}
\]

\[
\begin{array}{ll}
\text{find} \quad \xi_d = (x_d(\cdot), y_d(\cdot)) \in C^r \times L_{\infty}, r > 1 \quad \text{s.t.} \quad \eta_d \in \mathcal{T}, \ y_d(\cdot) = h(x_d(\cdot))
\end{array}
\]

\[
\begin{array}{ll}
\text{solve} \quad \text{minimize} \quad h_{c_e, \rho}(\xi) = h(\xi) + \epsilon_c b_{k_e, \rho}(\xi)
\end{array}
\]

\[
\begin{array}{ll}
\text{subj. to} \quad \xi = \mathcal{P}(\xi)
\end{array}
\]

\[
\begin{array}{ll}
\text{or the equivalent unconstrained version}
\end{array}
\]

\[
\begin{array}{ll}
\text{minimize} \quad g(\xi) = h_{c_e}(\mathcal{P}(\xi))
\end{array}
\]

\[
\begin{array}{ll}
tune \quad Q, R, P_1 \quad \text{shaping parameters}
\end{array}
\]

\[
\begin{array}{ll}
\epsilon_c, \delta_c, \rho \quad \text{closeness parameters}
\end{array}
\]

We call \( Q, R \) and \( P_1 \) shape parameters because they are used to weight some portions of the trajectory more than others. In our strategy we penalize the output much more than the other states and the inputs. We impose \( Q > 0, R > 0 \) and \( P_1 > 0 \) to have second order sufficiency condition local minimizers. The closeness parameters are used to set the operating conditions (\( \rho \)) and to tune the level of closeness of the feasible trajectory from the boundary of the feasible region (\( \epsilon_c, \delta_c \)).

Remark 4.1. It is important to underline that our goal is not to find the closest trajectory satisfying the operating conditions, that is we do not want to solve a constrained optimal control problem. We follow an engineering point of view, in the sense that we want a tool that provides feasible trajectories and where the “closeness” can be tuned by a “knob”.

Going back to the PVTOL, we set the following operating conditions on the control inputs parameterized by \( \rho \geq 1 \)

\[
\left( \frac{u_1 - \rho g}{\rho g} (1 + \frac{1}{\rho^2}) \right)^2 \leq 1
\]

and

\[
\left( \frac{u_2}{\mu_{2\max}} \right)^2 \leq 1
\]

where the nominal conditions are obtained for \( \rho = 1 \). Notice that starting with \( u_1(0) > 0 \) the physical bound of a positive thrust is also ensured.

4.3 Existence results

In the following we present results on the existence of solutions of the optimal control problem for suitable values of the operating conditions.

We consider the situation where the cost functional depends smoothly on a finite dimensional parameter and the system is independent of it. In the following, we will refer to this parameter as \( \rho \in \mathbb{R}^p \). This parameter may include, for instance, the scalar parameter \( \rho \) used for specifying the size of the feasible region as well as the scalar parameter \( \epsilon \) used in determining strictly feasible trajectories. We will suppose that the scaling and offsets of the parameters have been chosen in such a manner that the nominal value of the parameter vector is \( \rho = 0 \). We thus write

\[
g(\xi) = h(\mathcal{P}(\xi), \rho).
\]

Consider now the (local) minimization of \( g(\xi) \) as the parameter \( \rho \) is varied on a neighborhood of \( \rho = 0 \), where \( \xi_0 \) is known to be a second order sufficient condition (SSC) local minimizer of \( g_0(\xi) \). That is, \( \xi_0 \in \mathcal{T} \) is a local minimizer satisfying

\[
D^2_0 g_0(\xi_0) \cdot (\zeta, \zeta) \geq c_0 \|\zeta\|_{L_2}^2 \quad \text{for all } \zeta \in T_{\xi_0} \mathcal{T}.
\]

Since \( D^2_0 g_0(\xi) \) is continuous in both \( \xi \) and \( \rho \), we expect that, for each sufficiently small \( \rho \), there will be a corresponding SSC local minimizer \( \xi_\rho \) near \( \xi_0 \) and that the mapping \( \rho \mapsto \xi_\rho \) will be continuous, and perhaps differentiable. The key idea is to use an appropriate implicit function theorem (IFT) to solve the first order necessary condition equation

\[
Dg_\rho(\xi_\rho) = 0
\]

for \( \xi_\rho \) as a function of \( \rho \) starting from \( \xi_0 \) at \( \rho = 0 \). Proceeding formally, we differentiate (7) with respect to \( \rho \) to obtain

\[
\frac{\partial}{\partial \rho} Dg_\rho(\xi_\rho) + D\{Dg_\rho(\xi_\rho)\} \cdot \xi_\rho = 0.
\]
Thus, the derivative of $\xi_\rho$ with respect to $\rho$, $\xi'_\rho$, if it exists, is given formally by

$$\xi'_\rho = -[D\{Dg_\rho(\xi_\rho)\}]^{-1} \cdot \frac{\partial}{\partial \rho} Dg_\rho(\xi_\rho).$$

In this case, we might expect that there is an implicit function theorem that says something like, if $D\{Dg_\rho(\xi_\rho)\}$ is invertible at $\rho = 0$, then there is a neighborhood of $\rho = 0$ on which $\rho \mapsto \xi_\rho$ is well defined and $C^1$. In what sense should the operator $D\{Dg_0(\xi_0)\}$ be invertible and how can it be ensured? It turns out that the appropriate condition is that $D^2g_0(\xi_0)$ be strongly positive on $T_{\xi_0}T$, i.e., that it satisfy (6). The result is stated formally in the next theorem.

**Theorem 4.2.** Suppose that $\xi_0 \in \mathcal{T}$ is an SSC local minimizer of $g_0(\xi)$. Then, there is a $\delta > 0$ such that, for each $\rho$ such that $\|\rho\| < \delta$, there is a local SSC minimizer $\xi_\rho$ of $g_\rho(\xi)$ near $\xi_0$. Furthermore $\rho \mapsto \xi_\rho$ is continuously differentiable.

The proof of the theorem together with a formal treatment of these results may be found in (Notarstefano, 2007).

5. SIMULATIONS

In this section we present the simulation results for the PVTOL with $\epsilon_{\text{PVTOL}} = 0.05$. We perform a barrel roll subject to the bounds on $u_1$ and $u_2$ defined in the previous section. The desired path is depicted in Figure 2 with a dashed line. The desired velocity profile, assigned on the path, is constant to $v_0 = 6.65$ m/s in the first and last flat portions of the path and goes smoothly to $v_0 = 10.15$ m/s in the central loop. The three pictures compare the desired path and inputs with the feasible ones respectively computed for $\rho = 4$, an intermediate value for which the constraints are only slightly violated, and $\rho = 1$ that provides the specified constraints. The optimization was performed by iterating the barrier function method starting with $\epsilon_0 = 1$ up to $\epsilon_\rho = 0.01$. The maximum error on $y(\cdot)$ and $z(\cdot)$ is found to be 0.5 m. The result is quite surprising considering the tight limit on the thrust and roll moment. As it can be seen in Figure 3 the control tends to hit the boundary for a larger interval of time than the one where the constraints are violated (thus working in a noncausal fashion) in order to compensate the missing availability of input effort in those regions.

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